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$\mathcal{N} = 2$ Gauge Theories and Systems with Fractional Branes

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Abstract

We analyse the Seiberg Witten curve describing the $\mathcal{N} = 2$ gauge theory dual to the supergravity solution with fractional branes. Emphasis is given to those aspects that are related to stringy mechanism known as the *enhancement*. We also compare our results with the features of the supergravity duals, which have been variously interpreted in the literature. Known aspects of the $\mathcal{N} = 2$ gauge theories seem to agree with the supergravity solution, whenever the two theories can be faithfully compared.

1 Introduction

Supergravity solutions dual to non-conformal gauge theories have been extensively discussed in the past years [1]-[9]. These backgrounds are typically plagued by IR singularities, but, for sensible solutions, it is expected that these singularities can be completely resolved. There are known examples of different stringy resolution mechanisms. The *enhancement* mechanism [10] is well suited to describe such resolution for $\mathcal{N} = 2$ models. It is the stringy counterpart of well-known effects in the Seiberg-Witten (SW) solution for $\mathcal{N} = 2$ gauge theories. In the $\mathcal{N} = 1$ context, there are examples of deformed conformal theories which are described via D3-branes expanded into five-branes by the dielectric effect [11]. These solutions involve extra brane-like sources. A remarkable example of a completely regular $\mathcal{N} = 1$ supergravity solution without extra sources has been constructed by Klebanov and Strassler (KS) in [8].

In this paper we focus on the physics of four dimensional $\mathcal{N} = 2$ gauge theories in the Coulomb phase. These models naturally arise in string theory, if one looks at the low energy physics of regular and fractional D3-branes in type IIB theory compactified on orbifolds like T_4/Z_2 . The supergravity solutions for regular and fractional D3-branes placed at one of the orbifold singularities have been extensively discussed in the literature [6], [12]-[15]. In particular, the result for the fractional D3-branes bears some similarities with the $\mathcal{N} = 1$ KS solution, namely logarithmic behaviour of the twisted fields and, more suggestive, a logarithmic decreasing R-R five-form flux. These features of the supergravity solutions have been interpreted, on the field theory side, either as a signal of a possible Seiberg duality in a $\mathcal{N} = 2$ context [14], or as the description of a *Higgsed* vacuum [15]. It is therefore worthwhile to examine the predictions of the SW solution for the gauge theories that seem naturally related to these supergravity solutions. In fact, it is interesting to see whether the main features of the supergravity solutions are compatible with the known facts on the field theory side. For the example we are interested in the SW curve can be constructed following [16]. This comparison seems particularly relevant because the type IIB setup with fractional branes is dual to the M-theory model with 5-branes, that is used to determine the SW curve.

Existence of new physics for $\mathcal{N} = 2$ gauge theories is certainly intriguing. However, at the level of our analysis, we find that known aspects of the $\mathcal{N} = 2$ gauge theories seem to agree with the supergravity solution, whenever the two theories can be faithfully compared. On the gauge theory side, instanton corrections are important and responsible for the *enhancement*. The study of such corrections may shed light on how the enhancement mechanism is actually implemented in string theory. In this particular example, the contribution to the effective action of instantons (corresponding on the string side to wrapped D1-branes) and higher derivatives terms are difficult to estimate. Corrections are indeed deeply interconnected with tensionless string phases of the background. Of

course, it would be interesting to have a better understanding of these corrections.

In Section 2, we briefly review some properties of fractional branes at an orbifold singularity. In Section 3, 4 and 5, we discuss the SW curve for the associated $\mathcal{N} = 2$ theories. The weak coupling expansion of the curve is given, which could be useful for explicit instanton correction computations. Section 4 is especially devoted to a discussion of the geometrical aspects of the enhançon in these systems. We will mainly focus on the origin of the moduli space of the $\mathcal{N} = 2$ gauge theory. The gauge theory enhançon is very similar to the original one discussed in [10]. Much of the novelty comes from the fact that the system is now defined on a torus in M theory. In Section 6, we compare the quantum field theory results with the dual supergravity solutions, which have been discussed in the literature. In the Appendix, a dictionary for applying AdS/CFT rules to these systems is given, with attention to few subtleties in the identification of the moduli/parameters.

2 Fractional branes and $\mathcal{N} = 2$ gauge theories

A large class of $\mathcal{N} = 2$ gauge theories can be engineered by means of physical and fractional D3-branes in orbifold compactifications of type IIB string theory. We consider the case R^4/Z_2 and choose the coordinates of R^4 to be (x_6, x_7, x_8, x_9) . Particularly important for our construction are the twisted NS-NS and R-R scalars (b, c) . They can be thought as the flux of the NS-NS and R-R 2-form along the vanishing 2-cycle hidden in the orbifold singularity¹: $2\pi b = \int_{S_2} B$, $2\pi c = \int_{S_2} C_2$. The perturbative orbifold has value $b = 1/2$ [17], while b zero or integer correspond to non-perturbative phases of the theory with tensionless strings. Add now D3-branes with world-volume (0123) at the point $x_6 = x_7 = x_8 = x_9 = 0$. There are two basic types of D3-branes in this theory: fractional and anti-fractional D3-branes². Fractional branes have charges $(b, 1/2)$ with respect to the untwisted R-R form $C_{(4)}$ and the twisted one $C_{(4)}^T$, respectively; anti-fractional branes have charges $(1 - b, -1/2)$. With a fractional and an anti-fractional D3-brane we can make a physical D3-brane, whose charge is $(1, 0)$. There are several complementary descriptions for fractional branes:

- In the perturbative construction of the orbifold, each brane at $x_i^{(0)}$, $i = 6, 7, 8, 9$ has an image in $-x_i^{(0)}$. A brane and its image make up a physical brane, which can be moved at an arbitrary point in R^4/Z_2 . For $x_i^{(0)} = 0$, a physical brane appears as a composite object and can be split in the plane (x_4, x_5) . The constituents of a physical brane are the two types of fractional branes corresponding to the two

¹We work with the convention $2\pi\alpha' = 1$

²The two types of D3-branes are mutually BPS. We use the name *anti-fractional* with an abuse of language, following the interpretation as wrapped D5-branes.

irreducible representations of the Z_2 action on the Chan-Paton factors. Charges and tensions of these objects can be determined by the orbifold construction [18] or the boundary state formalism [19].

- A fractional brane can be represented as a D5-brane wrapped on the collapsed two-cycle of R^4/Z_2 [20, 21]. Similarly an anti-fractional brane is an anti-D5-brane with a non-trivial gauge field living on it $\int_{S_2} F = 2\pi$ [20, 21, 19]. This representation is particularly useful when $b \neq 1/2$ and the perturbative description of the orbifold is not adequate. In general, the induced D3-charges are b and $(1 - b)$, while the tensions are $|b|$ and $|1 - b|$. For $b \in [0, 1]$, they satisfy the BPS condition.
- In a useful T-dual picture, the same system is described by D4-branes stretched between NS-branes in type IIA ³. We use standard notations [16, 23]. The direction x_6 is compactified on a circle of radius L . The two NS-branes have world-volume $(0, 1, 2, 3, 4, 5)$ and sit at $x_6 = 0$ and $x_6 = 2\pi bL$, with $x_7 = x_8 = x_9 = 0$. The fractional branes can be identified with the D4-branes stretched from the first to the second NS-brane, the anti-fractional branes with the D4-branes stretched from the second to the first. A fractional and an anti-fractional brane can join and give a physical D4-brane, which can move away in (x_6, x_7, x_8, x_9) .

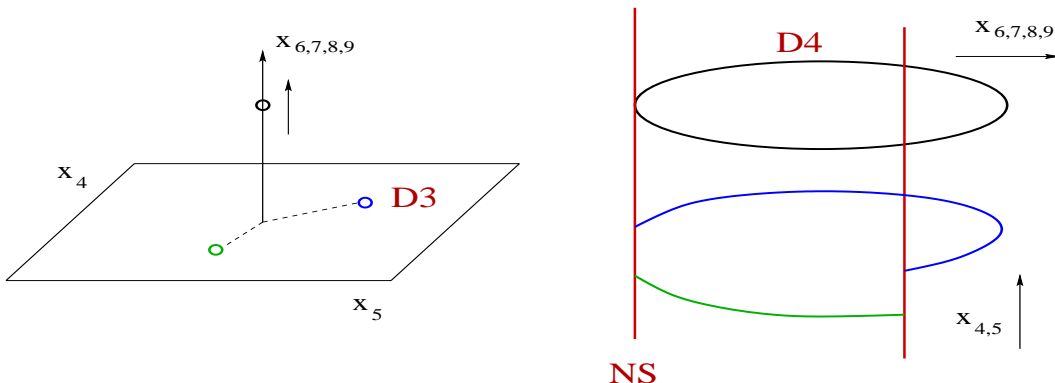


Figure 1: Type IIB and IIA picture for physical and fractional branes.

The $\mathcal{N} = 2$ gauge theory living on the D3-branes can be determined using the orbifold construction [18] or the brane rules [23]. For n_1 fractional and n_2 anti-fractional branes, the gauge theory is $U(n_1) \times U(n_2)$ with two bi-fundamental hypermultiplets. The gauge couplings of the two groups, τ_1, τ_2 , are determined in terms of the space-time fields by

$$\tau_1 = (b\tau + c), \quad \tau_2 = (1 - b)\tau - c, \quad (1)$$

³See, for example, [22] for a detailed discussion of the duality between the Hanany-Witten set-up and the fractional brane systems.

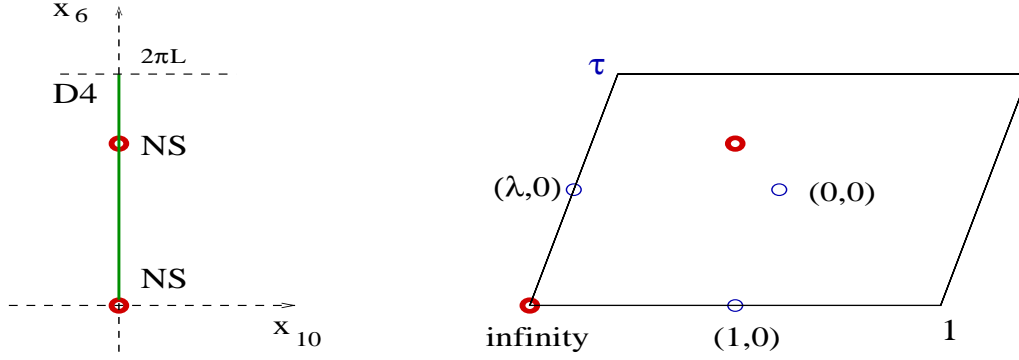


Figure 2: The fundamental domain for the torus E and its projection to a circle in type IIA with wrapped D4-branes. The four distinguished points $x = 0, 1, \lambda, \infty$ in the cubic representation are explicitly indicated.

where $\tau = C_0 + ie^{-\phi}$ is the complex dilaton of type IIB. The case $n_1 = n_2$ corresponds to a conformal field theory. The complex coupling constants of the two groups are exactly marginal parameters and the theory has an AdS dual: $\text{AdS}_5 \times S_5 / Z_2$ [24]. When $n_1 = N + M$ and $n_2 = N$, the theory is no more conformal and the coupling constants run at all scales. One of the two gauge factors is not asymptotically free and it is ill-defined in the UV. All the theories we are interested in can be obtained as suitable limits starting from the conformal case. We therefore analyse the SW curve for the conformal theory.

3 The SW curve

The curve for the conformal theory $SU(n) \times SU(n) \times U(1)$ with two bi-fundamental hypermultiplets was discussed in [16]. It was obtained by lifting the type IIA configuration with NS5 and D4-branes to M theory.

Call the M-theory coordinate x_{10} ($x_{10} \sim x_{10} + 2\pi R$). x_6 and x_{10} make a torus E in M theory, defined by (see Figure 2)

$$\begin{aligned} x_6 &\sim x_6 + 2\pi L, \\ x_{10} &\sim x_{10} + 2\pi \theta R. \end{aligned} \tag{2}$$

R plays no role in determining the holomorphic data in the SW curve and it is taken to be large so that we can use the semi-classical approximation of M theory. The modular parameter of E can be identified with the type IIB complex dilaton of the previous section. We describe the torus with a cubic equation,

$$y^2 = x(x-1)(x-\lambda), \tag{3}$$

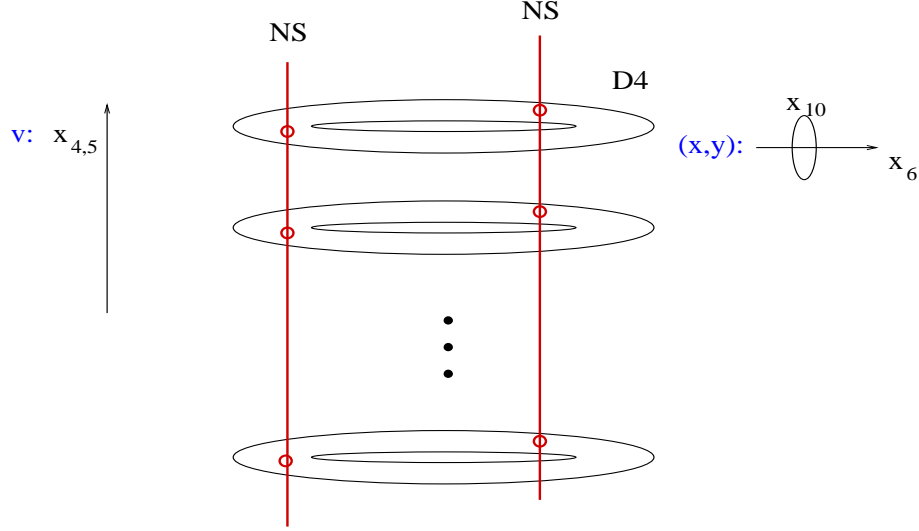


Figure 3: The SW curve as n -sheeted covering of the torus E .

where $\lambda = -\theta_2^4(\tau)/\theta_4^4(\tau)$.

The SW curve is an n -sheeted covering of the torus E , $F(x, y, v) = 0$, where F is a polynomial of degree n in v . The form of F was determined in [16]

$$F(x, y, v) = v^n + f_1(x, y)v^{n-1} + \dots + f_n(x, y). \quad (4)$$

Here f_i are meromorphic functions on E with simple poles at the positions of the NS-branes. Given the positions of the poles, each f_i depends on two arbitrary parameters. For each point $(x, y) \in E$, eq. (4) gives the positions of n branes in the complex plane $v = x_4 + ix_5$. F depends on $2n$ parameters which represent one mass parameter and the $2n - 1$ moduli for the Coulomb branch, roughly describing the positions of the fractional branes. The generic function f_i is of the form

$$f_i(x, y) = \frac{ax + by + c}{dx + ey + f}, \quad (5)$$

with parameters tuned in such a way that the two lines $ax + by + c = 0$ and $dx + ey + f = 0$ have a common intersection on E . The other two intersections are the two zeros and the two poles of the function. The f_i 's assume all the complex values twice, with four double points.

We can choose, for example,

$$f_i(x, y) = c_i + d_i \frac{y + y_B}{x - x_B}, \quad (6)$$

which has poles at P_∞ and $P_B = (x_B, y_B)$. As discussed in [16], c_1 can be interpreted as the $U(1)$ modulus and d_1 as related to the mass m of the hypermultiplets. With some

redefinitions, the curve becomes

$$\frac{R+S}{2} + \left(\frac{R-S}{2}\right) \frac{y+y_B}{x-x_B} = 0, \quad (7)$$

where R and S are polynomial of degree n normalized in such a way that $R, S = (v^n + \dots)$. If not explicitly stated, all polynomials in this paper are normalized so that the higher degree monomial has coefficient one. In this representation, the parameters in R and S are related to the classical positions of the two types of fractional branes: $R = \prod(v - z_i^{(1)})$ and $S = \prod(v - z_i^{(2)})$. Indeed for $R = S$ the curve factorizes into n copies of the torus E , describing n physical branes at arbitrary points (Figure 3). For $R \neq S$, the D4-branes are split and the NS-branes are bended. The actual meaning of the polynomials R and S in any corner of moduli space is determined by considering the appropriate limit of the curve.

For practical computations, it is sometimes convenient to map the problem to the parallelogram ($u \sim u+1, u \sim u+\tau$, see Figure 2). The variable u is immediately related to $x_6 + ix_{10}$. The poles are mapped to $u = 0$ (P_∞) and u_B (P_B). In terms of the brane construction outlined in the previous section, we have $u_B = b^{(0)}\tau + c^{(0)}$, where $b^{(0)}$ and $c^{(0)}$ are the asymptotic values of the twisted fields b, c . For simplicity, we will focus on the orbifold case $b^{(0)} = 1/2, c^{(0)} = 0$, where $\tau_1^{(0)} = \tau_2^{(0)} = \tau/2$. Moreover, we move the origin of the u -plane away from the position of the NS-brane by shifting $u \rightarrow u + \tau/4$. This will simplify the weak coupling expansion, where it is interesting to focus on one of the two gauge groups. Following [25], one can rewrite the meromorphic function $f = \frac{y+y_B}{x-x_B}$ as

$$f = \frac{\theta_3(u|\tau/2)}{\theta_4(u|\tau/2)} = \frac{\theta_3(2u|2\tau) + \theta_2(2u|2\tau)}{\theta_3(2u|2\tau) - \theta_2(2u|2\tau)}. \quad (8)$$

The curve then reads

$$R\theta_3(2u|2\tau) - S\theta_2(2u|2\tau) = 0. \quad (9)$$

The poles are now in $u = \tau/4$ and $u = 3\tau/4$. The curve is an infinite series in $t = e^{2\pi i u}$. As in [25], this infinite-degree polynomial in t specifies the positions in the complex u plane of an infinite number of NS branes. We will be interested in the weak coupling limit $\tau \rightarrow i\infty$. By truncating to the first order in $q = e^{2\pi i \tau}$ we have,

$$-q^{1/4}St + R - q^{1/4}S\frac{1}{t} = 0. \quad (10)$$

This is the curve for a gauge group $SU(n)$ with $2n$ flavours and coupling constant $\tau_1 = \tau/2$ [16, 26]. The moduli are specified by the zeros of the polynomial R . The masses for the flavours, which are equal two by two, are given by the zeros of the polynomial S . This completes the identification of R and S in the weak coupling limit. At each level

of approximation in q the curve is truncated to a degree k polynomial in t representing k NS-branes with n D4-branes stretched between them and two sets of n semi-infinite D4-branes on the right and on the left. There is a symmetry $t \rightarrow 1/t$ following from the symmetric choice of the poles. For example, at the next order we have

$$qRt^2 - q^{1/4}St + R - \frac{q^{1/4}S}{t} + \frac{qR}{t^2} = 0. \quad (11)$$

This is the curve for a $SU(n)^3$ theory, with the first and third factor identical.

Non conformal theories can be obtained by considering suitable limits in the moduli space. Consider, for example, M anti-fractional branes at z_∞ , M fractional branes and N physical branes at $z_i \ll z_\infty$ [14, 15]. We choose, for simplicity, a Z_M rotational invariant configuration for the M anti-D3-branes. We take therefore

$$R = P_{N+M}(v), \quad S = \bar{P}_N(v)(z_\infty^M - v^M), \quad (12)$$

with moduli in P_{N+M}, \bar{P}_N much smaller than z_∞ . For $|v| > z_\infty$ the theory is conformal. For $|v| < z_\infty$, the theory reduces at low energies to an $SU(N) \times SU(N+M)$ gauge theory with two bi-fundamentals. By matching the scales, we define the quantity $\Lambda^{2M} = z_\infty^{2M} e^{2\pi i \tau_1} = z_\infty^{2M} q^{1/2}$ appropriate for the IR strongly interacting theory $SU(N+M)$. The $SU(N)$ factor is IR free, but its dynamics can be slightly modified by the coupling to the other group. Since we are interested in the comparison with supergravity, we take the t'Hooft limit with $x = Ng_s$ and $y = Mg_s$ fixed. In this limit, $\Lambda \sim z_\infty e^{-\pi/2y}$ is kept fixed. In order to decouple the cut-off z_∞ , the appropriate limit is $z_\infty \rightarrow \infty, y \rightarrow 0$ with Λ fixed. Since the $SU(N)$ factor is not asymptotically free, the $SU(N) \times SU(N+M)$ theory is ill-defined in the UV. For this reason, we will keep a finite cut-off in the following. In the large M, N limit, the effects of the cut-off manifest very sharply near $v \sim z_\infty$.

Given the curve in the form (9), the computation of the first instantonic corrections in the weak coupling limit of the theory $SU(n) \times SU(n)$ could be explicitly carried out using the results in [25, 27].

4 The geometrical picture of the *enhancement*

The $\mathcal{N} = 2$ theory has moduli both for the physical and fractional branes. For every physical brane at \bar{v} , the polynomials R and S have a common factor $(v - \bar{v})$, which factorizes in the curve. Factorization of the curve is a signal of the singularity corresponding to the Higgs branch. However, physical branes do not bend the NS branes. This is the reason why physical branes will not affect most of our arguments.

The qualitative behaviour of the coupling constants for the two groups is determined by the *positions* of the two NS branes on the torus as a function of v [16]. These are

determined by the solutions of eq. (9) for fixed v . Consider, for simplicity, a configuration with M fractional and N physical branes at $v = 0$. For $|v| < z_\infty$ the exact curve (9) can be expanded

$$f(u, \tau) = \frac{S + R}{S - R} \sim 1 + 2 \left(\frac{v}{z_\infty} \right)^M. \quad (13)$$

The two relevant solutions for u can be obtained by expanding f for small q , $f \sim 1 + 2q^{1/4}(t + 1/t)$. Eq. (13) simplifies to

$$t + \frac{1}{t} \sim \left(\frac{v}{\Lambda} \right)^M. \quad (14)$$

If this condition is satisfied all higher order terms in f are negligible.

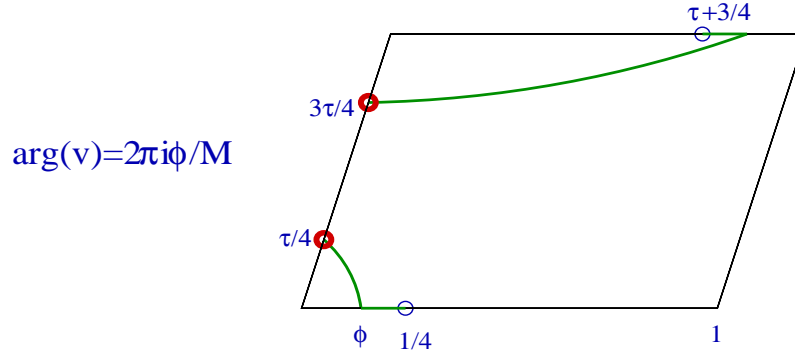


Figure 4: Positions of the two NS branes on the torus as functions of v . The argument of the complex number v has been fixed.

Eq. (14) is the same as the curve for pure $SU(M)$ with scale Λ . The only difference is that the two solutions of (14) should be brought back to the parallelogram. For $|v| > z_\infty$ and $|v| < \Lambda$, the positions of the NS-branes can be directly read from the exact curve (9), while for $z_\infty > |v| > \Lambda$ they are given by the roots of the quadratic equation (14). At leading order in M , one thus obtains (see also figure 4)

$$\begin{aligned} u_1(v) &= \tau/4, & u_2(v) &= 3\tau/4, & |v| > z_\infty \\ u_1(v) &= -\frac{M}{2\pi i} \log v/\Lambda, & u_2(v) &= \tau + \frac{M}{2\pi i} \log v/\Lambda, & \Lambda < |v| < z_\infty \\ u_1(v) &= 1/4, & u_2(v) &= \tau + 3/4, & |v| < \Lambda. \end{aligned} \quad (15)$$

As usual in the large M limit, corrections to these formulae rise up very sharply near z_∞ and Λ , and make the previous expressions completely smooth.

From the type IIA picture, we can roughly estimate the coupling constants for the two groups as $\tau_1(v) = u_1(v) - u_2(v) + \tau$ and $\tau_2(v) = u_2(v) - u_1(v)$. Note that $\tau_1 + \tau_2 = \tau$ is then automatically valid. We conclude that

$$\tau_1(v) = \frac{-2M}{2\pi i} \log v/\Lambda, \quad \tau_2(v) = \tau + \frac{2M}{2\pi i} \log v/\Lambda, \quad \Lambda < |v| < z_\infty. \quad (16)$$

This result agrees with the one-loop beta-function for the $SU(N) \times SU(N+M)$ theory. In the region $\Lambda < |v| < z_\infty$ instantonic corrections are indeed suppressed in the large N, M limit. They show up for $|v| \leq \Lambda$ where they force the branes in the positions $v = 1/4$ and $v = 3/4$. The previous qualitative argument suggests that, in the large M limit, $\tau_1(v)$ stops running at Λ : $\tau_1(v) = 1/2$ and $\tau_2(v) = \tau - 1/2$ for $|v| < \Lambda$. A negative imaginary part for τ_1 , as suggested by the perturbative result (16), would be unphysical.

Notice that, depending on the phase of v , the logarithmic behaviour of (15) seems to suggest that the NS branes can touch at $|v| = \Lambda$. Actually, they touch at exactly $2M$ points $v \sim \Lambda e^{2\pi i k/2M}$, $k = 0, 1, \dots, 2M-1$, close to a circle of radius Λ . These are the branch-points of the approximate curve (14) and they do not signal any singularity, since the curve is completely regular there. They give a reliable picture of the enhançon mechanism, as in the original example [10]. When the $\mathcal{N} = 2$ system is realized with branes, even if all the branes are *classically* at the origin (the *classical* QFT VEV's are zero), in the quantum theory they are disposed on a circle of radius Λ . If we send in another fractional brane, the classical value v covers the complex plane, but quantum mechanically the brane dissolves in the enhançon when $v \sim \Lambda$ and never enters the region $|v| < \Lambda$ [10]. This is easily seen by considering the curve for $M-1$ fractional branes classically at the origin and one at $v = \phi$: $P_{N+M} = v^{N+M-1}(v - \phi)$. As in [10], for $|\phi| > \Lambda$ there are $2M-2$ branch-points close to the circle $|v| = \Lambda$ and two at $v \sim \phi$, corresponding to a brane moving outside the original enhançon. For $|\phi| < \Lambda$ there are $2M$ branch-points close to the circle $|v| = \Lambda$, corresponding to the brane dissolved in the enhançon.

This discussion focused on the low-energy regime and it is appropriate to describe the group $SU(N+M)$ living on the fractional branes. Anti-fractional branes can be included in this geometrical picture of the enhançon by considering the exact curve (9). The meromorphic function (8) has double points at $u = 0, 1/2, \tau/2, (\tau+1)/2$. These points are the analogue for the exact curve of the coincident solutions for t in the approximate equation (14). Thus they can be interpreted as points where the NS-branes touch. Focusing on the double points on the torus, in the v -plane one can find the values of the branch points. The NS-branes touch for exactly $4M$ values of v , which have to be related to a total of M fractional and M anti-fractional branes. The branch-points at $u = 0, 1/2$

$$\frac{R+S}{S-R} = f(0), f(1/2) \quad \rightarrow \quad v \sim \Lambda e^{2\pi i k/2M}, \quad k = 0, \dots, 2M-1, \quad (17)$$

have been already discussed and correspond to fractional branes disposed on a circle of radius Λ . It is possible to verify that the new branch-points at $u = \tau/2, (\tau+1)/2$ determine the $2M$ values

$$\frac{R+S}{S-R} = f\left(\frac{\tau}{2}\right), f\left(\frac{\tau+1}{2}\right) \quad \rightarrow \quad v \sim z_\infty e^{2\pi i k/2M}, \quad k = 0, \dots, 2M-1, \quad (18)$$

corresponding to the anti-fractional branes disposed at circle at z_∞ . We can also verify that, if we send in an anti-fractional brane, it can move freely in the region $|v| < z_\infty$, as expected for a constituent of an IR free gauge group. If we replace $S \rightarrow v^N(v^{M-1} - z_\infty^{M-1})(v - \bar{\phi})$ in the previous formulae we find from eq. (18) $2M - 2$ branch-points at $|v| = z_\infty$ and two at $|v| \sim \bar{\phi}$, for every value $\bar{\phi} < z_\infty$. This corresponds to an anti-fractional probe that is free to move everywhere, even below the enhançon scale. What is amusing is that eq. (17) predicts in addition two extra branch-points at $|v| \sim \bar{\phi}$ if $\bar{\phi} < \Lambda$. It looks like a fractional brane has been unchained and follows the anti-fractional branes below the enhançon. This has a natural interpretation: If we start moving the cut-off branes below the scale Λ , the enhançon should be deformed and gradually disappear. Indeed, without cut-off there would be no asymptotically free group at low-energies and thus the enhançon is not needed!

We obtained a simple geometrical picture of the enhançon mechanism by taking sections of the curve at various symmetric points on the torus. It is clear that this interpretation can not be literally translated in terms of the D-brane setup, since close to the enhançon locus the geometry is subtle [28]. However, the field theory analysis suggests that at the scale Λ the dynamics of all kind of the D-branes is heavily affected by instanton corrections.

5 A different limit

In this Section, we consider a different scaling limit, slightly outside the purposes of our paper. This may serve as a consistency check of the discussed SW curve. In the limit where the coupling constant of one of the two gauge factors is finite, while the other goes to zero ($\tau_2 \rightarrow i\infty$, τ_1 fixed), we should recover the curve for the conformal $SU(n)$ theory with $2n$ flavours. In the type IIA picture, it can be accomplished by sending the radius L of the x_6 circle to infinity. We have two NS-branes with n D4-branes stretched in between, and two sets of n semi-infinite D4-branes on the right and on the left at the same position in $v = x_4 + ix_5$. This limit requires $b \rightarrow 0$, thus it is natural to expect a breakdown of the supergravity approximation. The limit is conveniently studied using a different representation of the meromorphic function than eq. (6). The torus (3) has a Z_2 automorphism $y \rightarrow -y$ with four fixed points at the values $x = 0, 1, \lambda, \infty$. On the natural variable for the fundamental domain pictured in Figure 2, the automorphism acts as $u \rightarrow -u$. Without loss of generality, we can choose the poles of the meromorphic function to be (y_B, x_B) and $(-y_B, x_B)$ so that the NS-branes are in a Z_2 invariant position

$$f_i = a_i + \lambda \frac{b_i}{x - x_B}. \quad (19)$$

All the points on E where f_i assumes the same complex value are then paired by Z_2 . In particular, $x = 0, 1, \lambda, \infty$ are the four double points. With obvious redefinitions, the curve can be written as

$$(T + V) + \lambda \frac{(T - V)}{x - x_B} = 0, \quad (20)$$

where T and V are polynomials of degree n . Take $\tau \rightarrow i\infty$ (which corresponds to $\lambda \rightarrow 0$) and rescale $x \rightarrow \lambda x, y \rightarrow \lambda y$ with $x_B = \lambda \hat{x}_B$. We have

$$\begin{aligned} y^2 &= -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}, \\ (T + V) &= \frac{(T - V)}{x - \hat{x}_B}. \end{aligned} \quad (21)$$

By eliminating x , we obtain the known curve for $SU(n)$ with $2n$ flavours [16, 26],

$$t^2 = P^2 + f^2 Q^2, \quad (22)$$

where

$$\begin{aligned} Q &= \frac{T + V}{2}, \\ P &= \frac{(\hat{x}_B + 1/2)V + (\hat{x}_B - 3/2)T}{2\hat{x}_B - 1}, \\ f^2 &= -(2\hat{x}_B - 1)^{-2}, \\ t &= f(R + S)y. \end{aligned} \quad (23)$$

Here $P = \prod_i^n (v - u_i)$ and $Q = \prod_i^n (v - m_i)$ are the degree n polynomials determining the $SU(n)$ moduli and the masses of the flavours, respectively. The SW differential $v dx/y$ on E indeed has poles at the masses m_i with residues proportional to m_i , while f determines the surviving coupling constant [16, 26], since, at weak coupling, $f \sim e^{\pi i \tau_1} \rightarrow 0$.

The CFT $SU(n)$ with $2n$ flavours is not easily obtained in the AdS/CFT correspondence, using supergravity only. One obvious problem is the large global symmetry. Another point signalling the breakdown of the supergravity approximation is related to the form of the conformal anomaly, which is usually written in terms of two coefficients a and c . In the theory under consideration a and c are not equal already at leading order in n . Thus supergravity, which always requires $a=c$, is not enough to describe such CFT. The type IIB orbifold with $b \rightarrow 0$ we are using is indeed a *stringy* background. On the other hand, we notice that quantum field theory instantons are mapped to D1 instantons in the string background. The contribution of D1 instantons survives in the scaling limit where τ_1 is kept fixed. The instanton moduli space of the theory $SU(n)$ with $2n$ flavours in the large n limit was studied in [29]. The result $\text{AdS}_5 \times S^1$ is appropriate for D1-instantons, which are localized on the fixed plane.

6 Comparison with the supergravity solution and discussion

The supergravity solution corresponding to $\mathcal{N} = 2$ fractional branes has been extensively discussed in [6, 13, 14, 12]. We consider all the branes in $x_6 = x_7 = x_8 = x_9 = 0$ and arbitrarily distributed in the (x_4, x_5) plane. It is convenient to introduce the complex variable $z = x_4 + ix_5$ and to denote the positions of the fractional and anti-fractional branes by $z^{(1)}, z^{(2)}$, respectively. In the gauge theory these correspond to VEV's of the Cartan values of the adjoint scalars parameterizing the generic vacuum.

Following [6, 30] we define⁴,

$$\gamma = 2\pi(c + \tau(b - 1/2)). \quad (24)$$

The supergravity equations of motion require an holomorphic γ . The linearized result [6],

$$\gamma(z) = \gamma^{(0)} + 2i \left(\sum_{i=1}^{n_1} \log(z - z_i^{(1)}) - \sum_{i=1}^{n_2} \log(z - z_i^{(2)}) \right). \quad (25)$$

combined with a black D3-brane ansatz

$$\begin{aligned} ds^2 &= Z^{-1/2} dx_\mu dx^\mu + Z^{1/2} ds_K^2, \\ F_5 &= dC_4 + *dC_4, \quad C_4 = \frac{1}{Z} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \quad (26)$$

is a solution of type IIB equations of motion provided that

$$-\square_K Z = \rho_{D3}(x) + \text{const } |\partial\gamma(z)|^2 \delta^{(4)}(x_6, x_7, x_8, x_9). \quad (27)$$

Here $\rho(x)$ is an arbitrary density of physical D3-branes [12, 13]. The general solution of this equation is

$$Z(x_T, z) = \sum_{i=1}^{n_1} \frac{b^{(0)}}{(x_T^2 + |z - z_i^{(1)}|^2)^2} + \sum_{i=1}^{n_2} \frac{1 - b^{(0)}}{(x_T^2 + |z - z_i^{(2)}|^2)^2} + \text{const} \int d^2w \frac{|\partial\gamma(w)|^2}{(x_T^2 + |z - w|^2)^2}. \quad (28)$$

The logarithmic behaviour in (25) reproduces the one-loop beta function of the $\mathcal{N} = 2$ gauge theory [6]. It is interpreted in the T-dual picture as the bending of the NS due to the D4-branes.

The solution (28) presents various kinds of singularity. There is certainly an IR singularity and, more generally, we expect singularities at the positions of the constituent

⁴Our conventions are slightly different from the ones of these papers. Here (b, c) have periods normalized to 1; moreover, we use the opposite sign for the Chern-Simons terms in the definition of the R-R field strength.

branes. In $\mathcal{N} = 2$, the enhançon mechanism [10] is usually invoked: The branes should resolve the singularity by forming shells. The enhançon mechanism thus suggests a natural IR cut-off for the integral in eq. (28). Notice also that the warp factor is singular for $x_T = 0, z = w$, if $\partial\gamma(w) \neq 0$. This can be interpreted as the result of the break down of the supergravity approximation near the orbifold fixed planes. Only if $\partial\gamma(w)$ has compact support, the solution is asymptotically ($z \gg 1$) well defined for all x_T .

For supergravity and AdS/CFT purposes, it is better to move together large bunches of branes. We therefore consider the limit $N \gg M \gg 1$. As usual, we keep the t'Hooft parameters $x = Ng_s$ and $y = Mg_s$ fixed. Moreover, the choice of a $U(1)$ invariant configuration helps in improving the UV behaviour of eq. (28)⁵. For these reasons, we choose the configuration analysed in Section 3 [14, 15]: We take M fractional and N physical branes at $z = 0$, and M anti-fractional branes in a rotational invariant configuration at $|z| = z_\infty$.

The one-loop beta functions are, for generic $b^{(0)}, c^{(0)}$,

$$2\pi\tau_1 = 2iM \log \frac{z}{\Lambda}, \quad \Lambda = z_\infty e^{2\pi i(c^{(0)} + \tau b^{(0)})/2M}, \quad (29)$$

and

$$2\pi\tau_2 = -2iM \log \frac{z}{\Lambda_2}, \quad \Lambda_2 = z_\infty e^{2\pi i(c^{(0)} + \tau(b^{(0)} - 1))/2M}, \quad (30)$$

where the Λ 's are the dynamically generated scales. We explicitly took the limit $N \gg M \gg 1$ and considered $|z| < z_\infty$. Notice that Λ_2 , which, according to this semi-classical reasoning, is the scale where the anti-fractional brane theory becomes strongly coupled, is above the cut-off and therefore it is not relevant to our analysis. Equations (29),(30) reduce to eq. (16) for $b^{(0)} = 1/2, c^{(0)} = 0$.

An important point, worth to be stressed, is that the one-loop behaviour (29),(30) is appropriate for not one but many points in moduli space. The presence of physical branes at arbitrary points, for example, does not change the result (29),(30).

In the limit $N \gg M \gg 1$, the solution in (28) resembles the KS solution [8] for $\mathcal{N} = 1$. This is mainly due to the log in eq. (25) and may suggest new physics for $\mathcal{N} = 2$, with a cascade mechanism similar to that in [8]. Actually, eqs. (1),(29) indicate that the background passes many times through values where $b \in \mathbb{Z}$ and non-perturbative phenomena become relevant [14, 15]. Start, for simplicity, at the cut-off with $b^{(0)} = 1/2, c^{(0)} = 0$. $b(z)$ decreases with $|z|$ and reaches the value $b(z) = 0$ for $|z| = \Lambda$. A tensionless string phase in type IIB requires $b = c = 0$, which (see eq. (1)) is the same as $\tau_1 = 0$. As also noticed in [15], this only selects $2M$ distinguished points on the circle $|z| = \Lambda$. They coincide with the branch-points for the curve discussed in Section 3 and represent

⁵For example, it improves the convergence of the spherical harmonic expansion of the integral in eq. (28).

an *enhancement*. The tension of a fractional brane probe is given by $\tau_1(z)$ and vanishes at the enhancement, suggesting that the probe cannot move below the scale $|z| = \Lambda$ [14, 15]. An anti-fractional probe can instead move even below $|z| = \Lambda$, since its tension is given by $\tau_2(z)$, which is non-vanishing. This perfectly agrees with our discussion based on the SW curve in Section 3.

The basic puzzle about the supergravity solution regards the scale where corrections actually start modifying it. Instantonic and higher derivative corrections are particularly complicated in these systems because of the presence of many other effects, for example tensionless string phases.

The crucial question is what happens below $|z| = \Lambda$. An extrapolation of the log behaviour would suggest not one, but many enhancements, whose physics needs to be explained, for example via a Seiberg duality for $\mathcal{N} = 2$ [14]. The phenomena discussed in [14] could in principle apply to some configurations, but it is difficult to make more precise statements. We focused on the origin of moduli space. If we take the attitude that the string resolution mechanism in type IIB should be the same as the one suggested by the SW curve, discussed in Section 3, we would conclude that there is only one enhancement at Λ . This is the point where tensionless strings and non-perturbative phenomena may become relevant and modify the semi-classical background. From the SW curve, we could expect that the supergravity fields for $|z| < \Lambda$ are frozen at the value they attained at the enhancement. The picture would be similar to the first example of enhancement discussed in [10]. The presence of physical branes should not modify the physics too much, at least when they sit at the origin of moduli space. In contrast with $\mathcal{N} = 1$, here there is a moduli space both for physical and fractional branes. We could have constructed our system by first sending N physical branes to the origin and only then trying to send in the M fractional ones. A reasonable expectation is that they form a spherical shell very similar to that for pure $SU(M)$ theory.

A standard observation in favour of the existence of many enhancements made in [14, 15] is that composite probes with both physical and fractional charges should be able to move below Λ and stop at a successive enhancement, the point where their tension vanishes. Take for example a probe with charges $(3/2, 1/2)$ in the orbifold background ($b = 1/2$). We can realize it by considering a bound state of a physical and a fractional brane. Notice that this system has many moduli: A fine tuning is required to move it as a single composite object. The log behavior suggests that its tension is finite at Λ and vanishes at a scale $\bar{z} < \Lambda$, which may define the second enhancement. However, as noticed in [14], the system stops being BPS at Λ . We can interpret this phenomenon as the fact that the fractional brane is obliged to stop at Λ , while the remaining physical brane is free to move below and reach the origin. This is consistent with our previous discussion.

Let us conclude this Section by discussing the decoupling limit $z_\infty \rightarrow \infty$. If $N = 0$ we

obtain the pure $SU(M)$ theory, which, as a quantum field theory, is defined at all scales. From eq. (29), we see that the correct limit for removing the cut-off is $z_\infty \rightarrow \infty$, $Mg_s \rightarrow 0$. Since, for $N = 0$, the t'Hooft parameter Mg_s has to be large for supergravity to be valid, this limit necessarily involves a string theory description. As usual, this was expected since the theory is asymptotically free in the UV.

For $N \neq 0$ we have a second option [14, 15]. We can take $x = Ng_s$ finite and large, so that all curvatures in the solution are small, while keeping $y = Mg_s$ fixed. In the decoupling limit $z_\infty \rightarrow \infty$, $y \rightarrow 0$, we obtain the theory $SU(N+M) \times SU(N)$. The scale for the asymptotically free factor $SU(N+M)$ is Λ . However, the theory is not well-defined since one of the gauge factors is not asymptotically free. The point Λ_2 , which signals non-perturbative effects for the $SU(N)$ gauge group, is now sent all the way to infinity. We expect that the one-loop behaviour is accurate for $z > \Lambda$, but fails in the UV, where new degrees of freedom are required for the QFT. This could be signaled, on the supergravity side, by D1-instanton effects.

In this paper, we mainly considered the origin of moduli space of the $\mathcal{N} = 2$ theory. In such a point, the presence of physical branes does not affect most of the reasonings. Other points are certainly interesting. It was suggested in [15] that a log solution with multiple enhançons is the description of a point of the moduli space with distributed physical branes. They should explain the logarithmically varying five-form flux. The large N limit considered in [15] seems to be different from that considered in this paper.

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Appendix: Comments on the AdS/CFT interpretation

In this Appendix we give a small dictionary for the AdS/CFT correspondence applied to our system, and discuss few subtleties. For the conformal case $SU(n) \times SU(n)$, the supergravity solution (26) is well defined and becomes $AdS_5 \times S_5/Z_2$. One can then use AdS/CFT to describe the conformal field theory corresponding to the origin of moduli space, where Higgs and Coulomb branch meet. We could also use the standard rules of AdS/CFT to study the Coulomb branch.

We first discuss the $U(1)$'s behaviour in the various pictures used in the paper. The

system of n fractional and anti-fractional D3 branes stuck at a Z_2 orbifold point in type IIB has a moduli space which is isomorphic to the Coulomb branch of an $\mathcal{N} = 2$, $U(n) \times U(n)$ gauge theory. The diagonal $U(1)$ factor is decoupled and corresponds to the center of mass motion of the system. $U(1)$'s factors may disappear in different representations and limits. The corresponding moduli are frozen and may reappear as mass parameters in the gauge theory⁶. In the picture with D4 and NS-branes, $U(1)$ factors are usually frozen [16]. In elliptic models, the diagonal $U(1)$ is present and decoupled, while the second $U(1)$ is frozen because $m = \sum z_i^{(1)} - \sum z_i^{(2)}$ is not normalizable [16]. The centers of mass of the two sets of D4-branes can be nevertheless at different points in $x_4 + ix_5$. m has to be interpreted not as a modulus but as a mass term for the hypermultiplets. In the AdS/CFT description of this system, all the $U(1)$ factors are, as usual, absent and the gauge group is $SU(n) \times SU(n)$. Supergravity solutions with non-zero m are interpreted as mass deformations of the $SU(n) \times SU(n)$ CFT. The Z_{2M} rotationally invariant configuration for anti-fractional branes in Sections 3 and 5 corresponds to choice of a point in the Coulomb branch. A configuration with all anti-fractional branes at a specific point z_∞ would correspond instead to a deformation of the CFT with a mass term combined with a choice of vacuum.

We now discuss the mapping between CFT operators and supergravity fields. For the untwisted fields, it follows from a Z_2 projection of the parent $\text{AdS}_5 \times S^5$ theory. For the twisted fields, the mapping was explicitly worked out in [30]. The order parameters for the Coulomb branch are associated with the operators

$$\begin{aligned} O_k &= \text{Tr}(b^{(0)}\phi_{(1)}^k + (1 - b^{(0)})\phi_{(2)}^k), \quad k = 2, 3, \dots \\ T_k &= \text{Tr}(\phi_{(1)}^k - \phi_{(2)}^k), \quad k = 2, 3, \dots \end{aligned} \tag{31}$$

O_k couple to the untwisted fields, specifically to the spherical harmonics of the metric. T_k couple to the harmonics of the twisted fields $\gamma(z) = \sum_n \gamma_n z^n, n \in Z$. As discussed in Section 2, the zero mode $n = 0$ corresponds to the dimension four operator $\text{Tr}(F_{(1)}^2 - F_{(2)}^2)$, dual to the coupling $\tau_1 - \tau_2$. The other harmonics correspond to the operators T_k and $W_i = \text{Tr}(F^2\phi_{(1)}^k - F^2\phi_{(2)}^k), k = 1, 2, \dots$. The only subtlety is that the $n = 1$ mode is associated with the dimension three operator $\int d^2\theta \text{Tr}(B_1 A_1 - B_2 A_2)$. T_1 is indeed identically zero. The absence of non-trivial dimension 1 operators in any conformal theory and a quick check to the mass spectrum found in [30] confirm this identification, which is consequence of the $U(1)$'s disappearing. The mapping operators/fields and the explicit coefficients in the definition (31) follow from the Born-Infeld action for wrapped D5-branes.

⁶This happens, as usual, because of the $\mathcal{N} = 2$ standard coupling $A\Phi B$ between the adjoint fields Φ (in $\mathcal{N} = 1$ notations) and the hypermultiplets (A, B) .

According to the standard AdS/CFT interpretation of supergravity solutions approaching AdS in the UV [31], the asymptotic behaviour of a supergravity field (with scaling dimension Δ) $\psi_O \sim z^{-\Delta}$ corresponds to a VEV for the corresponding operator O , while $\psi_O \sim z^{\Delta-4}$ indicates that the CFT is deformed with O . The UV expansion ($z \gg 1$) of equation (25) indeed suggests that the operators T_k have a VEV $\sum_i (z_i^{k(1)} - z_i^{k(2)})$, as expected for a generic point of the Coulomb branch. The leading term $\gamma \sim (z_i^{(1)} - z_i^{(2)})/z = m/z$ is appropriately interpreted as a deformation with a dimension three operator, rather than a VEV for a dimension one operator. As discussed above, this is the mass term $\int d^2\theta \text{Tr}(B_1 A_1 - B_2 A_2)$. A mass for hypermultiplets also requires a scalar mass term. We can find it in one of the harmonic for the metric, $Y_2(\hat{x}) = \sum_{a=6}^9 \hat{x}_a^2 - 2\hat{x}_4^2 - 2\hat{x}_5^2, \sum_{I=4}^6 \hat{x}^2 = 1$. For a correct AdS/CFT interpretation, such deformation with a dimension two operator should affect the expansion of the warp factor $Z \sim (1 + m^2 Y_2(\hat{x}) \log(r)/r^2)/r^4$ for large r . Qualitatively, this is easily extracted from the integral in eq. (28) for $\gamma \sim m/z$.

Finally, we could extract information about the VEV's of O_k from the large r expansion of eq. (28). The expansion of the first two terms in the right hand side is straightforward [31]. The third term requires particular care because of the integration over the entire plane z and the many types of divergences. Some assumptions and *renormalizations* are in general needed.

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